Journal of Statistical Physics, Vol. 105, Nos. 5/6, December 2001 (© 2001)

Constructive Approximations of Markov Operators

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Received June 19, 2000; revised May 2, 2001

We construct piecewise linear Markov finite approximations of Markov operators defined on $L^1([0,1]^N)$ and we study various properties, such as consistency, stability, and convergence, for the purpose of numerical analysis of Markov operators.

KEY WORDS: Markov operators; fixed densities; bounded variation; Markov finite approximations.

1. INTRODUCTION

In this paper we propose a class of continuous piecewise linear approximations to Markov operators defined on $L^1(I^N)$, where $I^N \equiv [0,1]^N$ is the N-dimensional unit cube of R^N, and we investigate various properties of such approximations. A unique feature of the analysis is that we can obtain the explicit constant for the stability of the numerical scheme, which is important for error estimates of computing fixed densities of Markov operators.

A bounded linear operator $P: L^1(I^N) \to L^1(I^N)$ is called a *Markov operator* if it is positive and preserves the L^1-norm of nonnegative functions. Markov operators are widely used in studying density evolution problems in partial differential equations, stochastic processes, discrete and continuous dynamical systems, and so on. In physical sciences densities are also employed for a stochastic description of the distribution of some physical quantities under the dynamical system.(1) An important subclass of Markov operators is the class of Frobenius–Perron operators in ergodic theory for

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0022-4715/01/1200-0863/0 © 2001 Plenum Publishing Corporation

finding absolutely continuous invariant measures (also called physical measures) of chaotic dynamics. A classic book on Markov operators is ref. 7, and a recent monograph by Lasota and Mackey$^{(12)}$ extensively studied various Markov operators and their applications in physical sciences, such as the approach of the Markov operators semigroup to the stochastic perturbation of discrete or continuous time systems and in particular to Fokker–Planck equations.

Because of the many applications of Markov operators in applied fields, their finite dimensional approximations are important in numerically solving the Markov operator equation $Pf = f$ for a fixed density, that is a fixed point of P which is also nonnegative with L^1-norm 1. Fixed densities usually describe the asymptotic statistical behavior of the underlying dynamical system. In developing efficient numerical methods it is desirable to preserve the physical structure of the operator. Usual numerical methods for solving operator equations, such as the Galerkin projection method and its variants, do not preserve the positivity of the Markov operator. In applications, however, we often require that a computed approximate fixed point be also a density. Thus, it is ideal to construct an approximate operator which is also a Markov operator of finite rank to guarantee the existence of an approximate fixed density according to the Frobenius–Perron theory for stochastic matrices. In this paper we present a numerical scheme of *piecewise linear Markov approximations* for a Markov operator $P: L^1(I^N) \to L^1(I^N)$. This scheme has an origin in Ulam's famous book$^{(17)}$ in which a piecewise constant approximation method was proposed which was extensively studied in Li's pioneering work$^{(14)}$ to prove its convergence for computing a fixed density of the Frobenius–Perron operator $P: L^1(0, 1) \to L^1(0, 1)$ associated with a piecewise C^2 and stretching mapping $S: [0, 1] \to [0, 1]$.

The Markov approximation scheme was first proposed in ref. 4 to improve the convergence rate of Ulam's original method, and was extended in ref. 6 for solving the fixed density problem of the Frobenius–Perron operator P corresponding to a piecewise C^2 and expanding mapping of the unit square $[0, 1]^2$ in the plane. Here we construct the scheme for a Markov operator on $L^1(I^N)$ for any N, and we will study various important properties for this class of approximations, in particular some stability result with an explicit constant will be established in terms of the variation norm.

Although Ulam's original piecewise constant method is also a structure preserving method, that is, the corresponding finite dimensional approximate operator maps densities to densities, its convergence rate under the L^1-norm, when applied to computing fixed densities of Frobenius–Perron operators for some classes of mappings, is only of order $O(\ln n/n)$.$^{(11, 15, 2)}$

Our method of piecewise linear approximations is shown to have the convergence rate of $O(1/n)$ even under the stronger BV-norm which will be defined below and is widely used in the convergence analysis of Markov operators.(4, 5) Therefore the scheme of piecewise linear Markov approximations studied in this paper seems an ideal numerical method for computing stationary densities of Markov operators, due to the above mentioned two facts of structure-preserving and fast convergence.

In the next section the Markov approximation will be introduced and some elementary properties will be given. In Section 3 a result on a uniform variation upper bound and the convergence in the BV-norm will be proved. Some application and numerical results will be given in Section 4.

2. PIECEWISE LINEAR MARKOV APPROXIMATIONS

In this paper we let $|x| \equiv \sqrt{x_1^2 + \cdots + x_N^2}$ denote the Euclidean 2-norm of a vector $x \in R^N$, and we let $\|f\| \equiv \int_{I^N} |f| \, dm$ denote the L^1-norm of $f \in L^1(I^N)$, where m is the Lebesgue measure. Let P: $L^1(I^N) \to L^1(I^N)$ be a Markov operator. Our purpose is to define a sequence of Markov operators P_n of finite rank that approximate P nicely. For this purpose we need to find a sequence of finite dimensional Markov operators that approximate the identity operator I.

Let the interval $I = [0, 1]$ be divided into n equal subintervals with length $h = 1/n$, and consequently the unit N-cube I^N is partitioned into n^N equal sub-cubes of volume h^N. Then each sub-cube is divided into $N!$ simplices of equal volume $h^N/N!$ in the standard way. Specifically, let x^0 with each component $x_i^0 < 1$ be a node of the partition and let σ be a permutation of $\{1, 2,..., N\}$. For $i = 1,..., N$ define x^i in succession by just adding h to the $\sigma(i)$th component of x^{i-1}. All such simplices $e = \text{conv}\{x^0, x^1,..., x^N\}$ constitute a standard triangulation T_n of I^N, which is sometimes referred to as *Kuhn's Triangulation.*(16) T_n consists of $n^N N!$ simplices $\{e_i\}$ with $(n+1)^N$ vertices $\{v_j\}$. It is well-known that T_n is a shape-regular and symmetric triangulation. In the following we let τ_v denote the number of the simplices of T_n with v as a vertex.

Lemma 2.1. Let $c = [x_1^0, x_1^0 + h] \times \cdots \times [x_N^0, x_N^0 + h]$ be a cube of the partition of I^N, and let $v = (x_1,..., x_N)$ be a vertex of c such that the number of i's with $x_i = x_i^0$ is l. Then the number of simplices of T_n in c is $l!\,(N-l)!$.

Proof. This is from the definition of Kuhn's triangulation and the fact that the number of permutations of $\{1,..., l\}$ and $\{l+1,..., N\}$ is $l!$ and $(N-l)!$, respectively. ∎

The number l in the lemma will be called the *relative number* of zeros of the vertex with respect to the starting vertex of the cube.

Proposition 2.1. Let r, s, t be nonnegative integers with sum N. Let $v=(x_1,\ldots,x_N)$ be a vertex of T_n such that $x_{i_1}=\cdots=x_{i_r}=0$, $x_{j_1}=\cdots=x_{j_s}=1$, and $0<x_{k_1},\ldots,x_{k_t}<1$. Then

$$\tau_v = N!\left[\frac{C_t^0}{C_N^r}+\frac{C_t^1}{C_N^{r+1}}+\cdots+\frac{C_t^{t-1}}{C_N^{r+t-1}}+\frac{C_t^t}{C_N^{r+t}}\right],$$

where $C_i^j=\frac{i!}{(i-j)!\,j!}$.

Proof. It is clear that v is a vertex of 2^t N-cubes of the partition of I^N, each of which is the Cartesian product of r intervals $[0,h]$, s intervals $[1-h,1]$, and t intervals of the type $[x_k-h,x_k]$ or $[x_k,x_k+h]$. Among them, there are C_t^0 cubes with v as a vertex of the relative number of zeros r, C_t^1 cubes with v as a vertex of the relative number of zeros $r+1,\ldots$, C_t^t cubes with v as a vertex of the relative number of zeros $r+t$. Thus, by Lemma 2.1,

$$\begin{aligned}\tau_v &= C_t^0 r!\,(N-r)!+C_t^1(r+1)!\,(N-r-1)!\\ &\quad+\cdots+C_t^{t-1}(r+t-1)!\,(N-r-t+1)!+C_t^t(r+t)!\,(N-r-t)!\\ &= C_t^0\,\frac{N!}{C_N^r}+C_t^1\,\frac{N!}{C_N^{r+1}}+\cdots+C_t^{t-1}\,\frac{N!}{C_N^{r+t-1}}+C_t^t\,\frac{N!}{C_N^{r+t}}\\ &= N!\left[\frac{C_t^0}{C_N^r}+\frac{C_t^1}{C_N^{r+1}}+\cdots+\frac{C_t^{t-1}}{C_N^{r+t-1}}+\frac{C_t^t}{C_N^{r+t}}\right]\end{aligned}$$ ∎

Remark 2.1. Since $i!\,(N-i)!\geqslant[\Gamma(N/2)]^2$ for $i=0,1,\ldots,N$, a lower bound of τ_v is $2^t[\Gamma(N/2)]^2$.

Corollary 2.1. Let v be an interior vertex of T_n. Then $\tau_v=(N+1)!$.

Proof. In this case, $r=0$, $s=0$, and $t=N$. ∎

Corollary 2.2. Let v be a relative interior vertex in the $N-r$ dimensional face $x_{i_1}=\cdots=x_{i_r}=0$ of I^N. Then

$$\tau_v = N!\left[\frac{C_{N-r}^0}{C_N^r}+\frac{C_{N-r}^1}{C_N^{r+1}}+\cdots+\frac{C_{N-r}^{N-r-1}}{C_N^{N-1}}+\frac{C_{N-r}^{N-r}}{C_N^N}\right]\geqslant N!.$$

Proof. Here $s=0$ and $t=N-r$. ▮

Corollary 2.3. Let v be a relative interior vertex in the $N-s$ dimensional face $x_{j_1}=\cdots=x_{j_s}=1$ of I^N. Then

$$\tau_v=N!\left[\frac{C^0_{N-s}}{C^0_N}+\frac{C^1_{N-s}}{C^1_N}+\cdots+\frac{C^{N-s-1}_{N-s}}{C^{N-s-1}_N}+\frac{C^{N-s}_{N-s}}{C^{N-s}_N}\right]\geqslant N!.$$

Proof. Now $r=0$ and $t=N-s$. ▮

Corollary 2.4. Let v be a vertex of I^N such that $x_{i_1}=\cdots=x_{i_r}=0$ and $x_{j_1}=\cdots=x_{j_{N-r}}=1$. Then $\tau_v=r!\,(N-r)!\geqslant[\Gamma(N/2)]^2$.

Proof. The result is true since $t=0$. ▮

Let Δ_n be the set of all continuous piecewise linear functions corresponding to the triangulation T_n. Then Δ_n is an $(n+1)^N$-dimensional subspace of $L^1(I^N)$. Denote by ϕ_i the unique element in Δ_n such that

$$\phi_i(v_j)=\delta_{ij},\qquad i,j=1,\dots,(n+1)^N,$$

where $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ otherwise. Then $\{\phi_i\}$ is the canonical basis for Δ_n. In fact for each $g\in\Delta_n$,

$$g=\sum_{i=1}^{(n+1)^N} g(v_i)\,\phi_i.$$

Now we define the operator $Q_n\colon L^1(I^N)\to L^1(I^N)$ by

$$Q_n f=\sum_{i=1}^{(n+1)^N} f_i\phi_i,$$

where for each i,

$$f_i=\frac{1}{m(V_i)}\int_{V_i} f\,dm=\frac{N!}{\tau_i h^N}\int_{V_i} f\,dm$$

is the average value of f over V_i. Here V_i is the union of all the τ_i simplices of T_n that have v_i as a vertex. Note that $\phi_i\geqslant 0$, its support supp ϕ_i of ϕ_i is V_i, and its L^1-norm

$$\|\phi_i\|=\frac{\tau_i h^N}{(N+1)!}$$

for each i. Moreover,

$$\sum_{i=1}^{(n+1)^N} \phi_i(x) \equiv 1.$$

Proposition 2.2. $Q_n: L^1(I^N) \to L^1(I^N)$ is a Markov operator.

Proof. It is clear that Q_n is a positive linear operator with range $R(Q_n) = \Delta_n$. Let $f \in L^1(I^N)$ be nonnegative. Then

$$\begin{aligned}\int_{I^N} Q_n f\, dm &= \sum_{i=1}^{(n+1)^N} f_i \int_{I^N} \phi_i\, dm \\ &= \sum_{i=1}^{(n+1)^N} \frac{N!}{\tau_i h^N} \int_{V_i} f\, dm \frac{\tau_i h^N}{(N+1)!} \\ &= \frac{1}{N+1} \sum_{i=1}^{(n+1)^N} \int_{V_i} f\, dm = \int_{I^N} f\, dm,\end{aligned}$$

where the last equality follows from the fact that each simplex has exactly $N+1$ vertices. ▮

Proposition 2.3. There is a constant C independent of n such that

$$\|Q_n f - f\| \leqslant Ch \int_{I^N} |\text{grad } f|\, dm, \qquad \forall f \in W^{1,1}(I^N), \tag{1}$$

where $\text{grad } f = (\partial f/\partial x_1, ..., \partial f/\partial x_N)^T$ is the gradient of f in the weak sense of Sobolev and $W^{1,1}(I^N)$ is the usual Sobolev space.

Proof. By (7.45) in ref. 8, for each i,

$$\int_{V_i} |f - f_i|\, dm \leqslant \left(\frac{\omega_N}{m(V_i)}\right)^{1-\frac{1}{N}} d_i^N \int_{V_i} |\text{grad } f|\, dm,$$

where $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ is the volume of the unit ball in R^N and $d_i = \text{diam } V_i$. Since V_i is contained in an N-cube of side $2h$ centered at v_i (in fact the N-cube is the union of the 2^N cubes of the partition of I^N with v_i as a common vertex), $d_i \leqslant 2\sqrt{N}\, h$. Also note that $m(V_i) = \tau_i h^N/N!$, so

$$\begin{aligned}\int_{V_i} |f - f_i|\, dm &\leqslant \left(\frac{\omega_N N!}{\tau_i h^N}\right)^{1-\frac{1}{N}} (2\sqrt{N}\, h)^N \int_{V_i} |\text{grad } f|\, dm \\ &= \left(\frac{\omega_N N!}{\tau_i}\right)^{1-\frac{1}{N}} (4N)^{\frac{N}{2}}\, h \int_{V_i} |\text{grad } f|\, dm.\end{aligned}$$

Because of (5),

$$|Q_n f(x)-f(x)| = \left|\sum_{i=1}^{(n+1)^N} f_i\phi_i(x) - \sum_{i=1}^{(n+1)^N} f(x)\,\phi_i(x)\right|$$
$$\leqslant \sum_{i=1}^{(n+1)^N} |f_i - f(x)|\,\phi_i(x).$$

Since supp $\phi_i = V_i$ and $\tau_i \geqslant (\Gamma(N/2))^2$, from the above

$$\begin{aligned}\|Q_n f - f\| &\leqslant \sum_{i=1}^{(n+1)^N} \int_{V_i} |f-f_i|\,dm\\ &\leqslant \sum_{i=1}^{(n+1)^N} \left(\frac{\omega_N N!}{\tau_i}\right)^{1-\frac{1}{N}} (4N)^{\frac{N}{2}}\, h \int_{V_i} |\text{grad } f|\,dm\\ &\leqslant \left[\frac{\omega_N N!}{\left(\Gamma\left(\frac{N}{2}\right)\right)^2}\right]^{1-\frac{1}{N}} (4N)^{\frac{N}{2}}\, h \sum_{i=1}^{(n+1)^N} \int_{V_i} |\text{grad } f|\,dm\\ &= \left[\frac{\omega_N N!}{\left(\Gamma\left(\frac{N}{2}\right)\right)^2}\right]^{1-\frac{1}{N}} (4N)^{\frac{N}{2}}\,(N+1)\, h \int_{I^N} |\text{grad } f|\,dm.\end{aligned}$$

Hence (1) is true with

$$C = \left[\frac{\omega_N N!}{\left(\Gamma\left(\frac{N}{2}\right)\right)^2}\right]^{1-\frac{1}{N}} (4N)^{\frac{N}{2}}\,(N+1). \quad \blacksquare$$

3. VARIATION INEQUALITIES

The modern notion of variation for functions of several variables is useful in many problems of dynamical systems and numerical analysis. In particular it has played an important role in the existence problem and numerical analysis of a class of Frobenius–Perron operators (see, e.g., refs. 10, 6 and references therein). In this section we will prove that the Markov finite approximations sequence Q_n defined in the previous section will satisfy a uniform variation inequality, and an explicit expression of the constant in the inequality is also available. This is a stability result under the variation norm. Moreover, we will strengthen Proposition 2.3 by proving a consistency result in the variation norm. Such results will be useful in the convergence analysis and error estimates of the Markov approximation method in solving Markov operator equations.

Definition 3.1.[9] Let Ω be an open set in R^N and $f \in L^1(\Omega)$. The number (including ∞)

$$V(f; \Omega) = \sup \left\{ \int_\Omega f \operatorname{div} w \, dm : w \in C_0^1(\Omega; R^N), |w(x)| \leqslant 1, x \in \Omega \right\}$$

is called the variation of f over Ω. If $V(f; \Omega) < \infty$, then f is said to be of bounded variation in Ω. $BV(\Omega)$ denotes the space of all functions in $L^1(\Omega)$ with bounded variation.

Note that $BV(\Omega)$ is a Banach space under the norm $\|f\|_{BV} \equiv \|f\| + V(f; \Omega)$, its closed unit ball is compact in $L^1(\Omega)$ if Ω is bounded with Lipschitz boundary, and the Sobolev space $W^{1,1}(\Omega)$ is a closed subspace of $BV(\Omega)$ with $V(f; \Omega) = \int_\Omega |\operatorname{grad} f| \, dm$ for $f \in W^{1,1}(\Omega)$. Some other properties are referred to ref. 9.

Lemma 3.1. Let e be a simplex in R^N with vertices $v_0, v_1, \ldots, v_N$ such that $|v_k - v_{k-1}| = h$ and $\{v_k - v_{k-1}\}$ are orthogonal to each other for $k = 1, \ldots, N$. If $g(x) = a^T x + b$ on e, then

$$a = \frac{1}{h^2} \sum_{k=1}^{N} [g(v_k) - g(v_{k-1})](v_k - v_{k-1}), \tag{2}$$

and thus,

$$V(g; e) = |a| \, m(e) = \frac{m(e)}{h} \left(\sum_{k=1}^{N} (g(v_k) - g(v_{k-1}))^2 \right)^{1/2}. \tag{3}$$

Proof. Since $g(v_k) = a^T v_k + b$ for $k = 0, 1, \ldots, N$,

$$a^T (v_k - v_{k-1}) = g(v_k) - g(v_{k-1}), \qquad k = 1, \ldots, N.$$

Now the $N \times N$ matrix

$$\frac{1}{h} [v_1 - v_0, \ldots, v_N - v_{N-1}]$$

is orthogonal, so (2) follows and (3) is immediate since $g \in W^{1,1}(e)$. ▮

Now we estimate the variation of $Q_n f$. First note that $Q_n f \in W^{1,1}(I^N)$ since $Q_n f \in \Delta_n$. Let e_i be the ith simplex of T_n, $i = 1,\ldots, n^N N!$, and let $Q_n f(x) = a_i^T x + b_i$ on e_i. Then

$$V(Q_n f; I^N) = \sum_{i=1}^{n^N N!} |a_i|\, m(e_i) = \frac{h^N}{N!} \sum_{i=1}^{n^N N!} |a_i|. \tag{4}$$

Denote by $v_0^i,\ldots, v_N^i$ the vertices of e_i ordered naturally such that $\{v_k^i - v_{k-1}^i\}$ are orthogonal to each other for $k = 1,\ldots, N$. For $k = 0,\ldots, N$ let $Q_n f(v_k^i) = q_k^i$ and let V_k^i be the union of all the τ_k^i simplices of T_n that have v_k^i as a vertex. Recall that each q_k^i is the average value of f over V_k^i. By (4) and Lemma 3.1,

$$\begin{aligned} V(Q_n f; I^N) &= \frac{h^{N-1}}{N!} \sum_{i=1}^{n^N N!} \left(\sum_{k=1}^{N} (q_k^i - q_{k-1}^i)^2 \right)^{1/2} \\ &\leqslant \frac{h^{N-1}}{N!} \sum_{i=1}^{n^N N!} \sum_{k=1}^{N} |q_k^i - q_{k-1}^i|. \end{aligned} \tag{5}$$

Theorem 3.1. There holds

$$V(Q_n f; I^N) \leqslant C_{BV} V(f; I^N), \qquad \forall f \in BV(I^N), \quad \forall n, \tag{6}$$

where the constant

$$C_{BV} = \frac{2N!\,(\omega_N N!)^{1-\frac{1}{N}}\,(4N+5)^{\frac{N}{2}}\,[(N+1)\,N-1]}{\left(\Gamma\left(\frac{N}{2}\right)\right)^{2(2-\frac{1}{N})}}.$$

Proof. First we assume that $f \in W^{1,1}(I^N)$. From (5), it is enough to estimate $|q_k^i - q_{k-1}^i|$. By (7.45) in ref. 8,

$$\begin{aligned} |q_k^i - q_{k-1}^i| &= \left| \frac{1}{m(V_k^i)} \int_{V_k^i} f\, dm - q_{k-1}^i \right| \\ &\leqslant \frac{1}{m(V_k^i)} \int_{V_k^i} |f(x) - q_{k-1}^i|\, dm(x) \\ &\leqslant \frac{1}{m(V_k^i)} \int_{V_k^i \cup V_{k-1}^i} |f(x) - q_{k-1}^i|\, dm(x) \\ &\leqslant \frac{1}{m(V_k^i)} \left(\frac{\omega_N}{m(V_{k-1}^i)} \right)^{1-\frac{1}{N}} d_{ik}^N \int_{V_k^i \cup V_{k-1}^i} |\text{grad } f|\, dm, \end{aligned}$$

where $d_{ik}=\operatorname{diam}(V_k^i\cup V_{k-1}^i)$. Since $m(V_k^i)=\tau_k^i h^N/N!$, $\tau_k^i\geqslant(\Gamma(N/2))^2$, and $d_{ik}\leqslant\sqrt{4N+5}\,h$,

$$\begin{aligned}|q_k^i-q_{k-1}^i| &\leqslant \frac{N!}{\tau_k^i h^N}\left(\frac{\omega_N N!}{\tau_{k-1}^i h^N}\right)^{1-\frac{1}{N}} d_{ik}^N \int_{V_k^i\cup V_{k-1}^i} |\operatorname{grad} f|\,dm\\ &\leqslant \frac{N!\,(\omega_N N!)^{1-\frac{1}{N}}\,(4N+5)^{\frac{N}{2}}}{h^{N-1}\left(\Gamma\left(\frac{N}{2}\right)\right)^{2(2-\frac{1}{N})}} \int_{V_k^i\cup V_{k-1}^i} |\operatorname{grad} f|\,dm.\end{aligned}$$

It follows that

$$\begin{aligned}V(Q_n f; I^N) &\leqslant \frac{h^{N-1}}{N!}\sum_{i=1}^{n^N N!}\sum_{k=1}^{N}\frac{N!\,(\omega_N N!)^{1-\frac{1}{N}}\,(4N+5)^{\frac{N}{2}}}{h^{N-1}\left(\Gamma\left(\frac{N}{2}\right)\right)^{2(2-\frac{1}{N})}} \int_{V_k^i\cup V_{k-1}^i} |\operatorname{grad} f|\,dm\\ &\leqslant \frac{(\omega_N N!)^{1-\frac{1}{N}}\,(4N+5)^{\frac{N}{2}}}{\left(\Gamma\left(\frac{N}{2}\right)\right)^{2(2-\frac{1}{N})}}\sum_{i=1}^{n^N N!}\sum_{k=1}^{N}\int_{V_k^i\cup V_{k-1}^i} |\operatorname{grad} f|\,dm.\end{aligned}$$

Since each $V_k^i\cup V_{k-1}^i$ contains at most $[2(N+1)!-2(N-1)!]\,N=2N!\,[(N+1)\,N-1]$ simplices, we see that

$$\sum_{i=1}^{n^N N!}\sum_{k=1}^{N}\int_{V_k^i\cup V_{k-1}^i} |\operatorname{grad} f|\,dm\leqslant 2N!\,[(N+1)\,N-1]\int_{I^N} |\operatorname{grad} f|\,dm.$$

Thus (6) is true in the case of $f\in W^{1,1}(I^N)$.

For an arbitrary $f\in BV(I^N)$, by Theorem 1.17 in ref. 9, there exists a sequence $\{f_j\}\subset C^\infty(I^N)$ such that

$$\lim_{j\to\infty}\|f_j-f\|=0$$

and

$$\lim_{j\to\infty}V(f_j; I^N)=V(f; I^N).$$

From the above proof, (6) is valid for each f_j. Since for each n we have $\lim_{j\to\infty}\|Q_n f_j-Q_n f\|=0$, using Theorem 1.9 of ref. 9, we have

$$\begin{aligned}V(Q_n f; I^N) &\leqslant \liminf_{j\to\infty} V(Q_n f_j; I^N)\\ &\leqslant \lim_{j\to\infty} C_{BV}V(f_j; I^N)=C_{BV}V(f; I^N). \quad \blacksquare\end{aligned}$$

Now we prove the strong convergence of the sequence $\{Q_n f\}$ to f under the BV-norm for sufficiently smooth f.

Theorem 3.2. As $h \to 0$, there hold

$$\|Q_n f - f\|_{BV} = O(h), \qquad \forall f \in C^2(I^N), \tag{7}$$

$$\|Q_n f - f\|_{BV} = o(1), \qquad \forall f \in W^{1,1}(I^N). \tag{8}$$

Proof. As before let e_i be the ith simplex of T_n with the naturally ordered vertices $v_0^i, \ldots, v_N^i$, $v_k^i = v_{k-1}^i + hu^{\sigma(k)}$, $k = 1, \ldots, N$, where $u^{\sigma(k)}$ is the $\sigma(k)$th canonical basis of R^N for some permutation σ of $\{1, 2, \ldots, N\}$, and let $Q_n f(v_k^i) = q_k^i$. Then, by Lemma 3.1,

$$\text{grad } Q_n f(x) = \sum_{k=1}^{N} \frac{q_k^i - q_{k-1}^i}{h} u^{\sigma(k)}, \qquad x \in e_i.$$

Thus, since $Q_n f - f \in W^{1,1}(I^N)$,

$$\begin{aligned} V(Q_n f - f; I^N) &= \sum_{i=1}^{n^N N!} \int_{e_i} |\text{grad}(Q_n f - f)(x)| \, dm(x) \\ &= \sum_{i=1}^{n^N N!} \int_{e_i} \left| \sum_{k=1}^{N} \frac{q_k^i - q_{k-1}^i}{h} u^{\sigma(k)} - \text{grad } f(x) \right| dm(x) \\ &= \sum_{i=1}^{n^N N!} \int_{e_i} \left| \sum_{k=1}^{N} \left(\frac{q_k^i - q_{k-1}^i}{h} - \frac{\partial f(x)}{\partial x_{\sigma(k)}} \right) u^{\sigma(k)} \right| dm(x) \\ &= \sum_{i=1}^{n^N N!} \int_{e_i} \sqrt{\sum_{k=1}^{N} \left(\frac{q_k^i - q_{k-1}^i}{h} - \frac{\partial f(x)}{\partial x_{\sigma(k)}} \right)^2} \, dm(x) \\ &= \sum_1 + \sum_2 \end{aligned}$$

where $\sum_1$ is the sum of the integrals over all simplices e_i all of whose vertices are interior to I^N and $\sum_2$ is the remaining sum. Let e_i be a simplex in $\sum_1$. Then for $k = 0, 1, \ldots, N$, from Taylor's expansion, we have

$$f(x) = f(v_k^i) + (\text{grad } f(v_k^i))^T (x - v_k^i) + O(|x - v_k^i|^2),$$

from which it follows that

$$q_k^i \equiv \frac{1}{m(V_k^i)} \int_{V_k^i} f(x) \, dm(x) = f(v_k^i) + O(h^2)$$

because $\int_{V_k^i} (\text{grad } f(v_k^i))^T (x - v_k^i)\, dm = 0$ due to the fact that V_k^i is symmetric about v_k^i. Hence for $k = 1, \ldots, N$,

$$\frac{q_k^i - q_{k-1}^i}{h} = \frac{f(v_k^i) - f(v_{k-1}^i)}{h} + O(h). \tag{9}$$

Since $v_k^i = v_{k-1}^i + hu^{\sigma(k)}$,

$$\frac{f(v_k^i) - f(v_{k-1}^i)}{h} = \frac{\partial f(v_k^i)}{\partial x_{\sigma(k)}} + O(h). \tag{10}$$

Combining (9) and (10), and using the fact that

$$\frac{\partial f(x)}{\partial x_{\sigma(k)}} = \frac{\partial f(v_k^i)}{\partial x_{\sigma(k)}} + O(|x - v_k^i|),$$

we get

$$\int_{e_i} \sqrt{\sum_{k=1}^{N} \left(\frac{q_k^i - q_{k-1}^i}{h} - \frac{\partial f(x)}{\partial x_{\sigma(k)}} \right)^2}\, dm(x) = O(h)\, m(e_i),$$

which implies that $\sum_1 = O(h)$. On the other hand, since the Lebesgue measure of the union of all the simplices in $\sum_2$ is of order $O(h)$ and since the integrand is bounded, $\sum_2 = O(h)$. Thus (7) is true. And (8) follows from (7), Theorem 3.1, and a density argument. ▮

Remark 3.1. Theorem 3.2 improves Proposition 2.3 under a mild smooth condition on f. It should be noted that although it satisfies (1), Ulam's method does not satisfy (7), which makes the piecewise linear Markov approximation method more appealing in the numerical computation related to Markov operators.

4. SOME APPLICATION

In this section we give an application of the main result. The general setting is that we want to calculate a fixed density of a Markov operator $P: L^1(I^N) \to L^1(I^N)$. Assume that P satisfies the property that there are two positive constants $\alpha < 1$ and β such that

$$\|Pf\|_{BV} \leqslant \alpha\, \|f\|_{BV} + \beta\, \|f\|, \qquad \forall f \in BV(I^N). \tag{11}$$

The importance of the above inequality for the existence of fixed densities of Markov operators was first indicated in the seminal paper of Lasota and

Yorke[13] for proving the existence of fixed densities of a class of Frobenius–Perron operators in ergodic theory, and then used, among others, in refs. 14, 10, 5, 6, and 15.

4.1. A Finite Element Method

Associated with the Kuhn triangulation T_n and the corresponding Markov approximation Q_n, we propose a finite element scheme for computing a fixed density of a Markov operator $P\colon L^1(I^N) \to L^1(I^N)$ as follows: Find a fixed density $f_n \in \varDelta_n$ such that

$$P_n f_n = f_n, \tag{12}$$

where $P_n \equiv Q_n P$. Since P_n is Markov operator from $\varDelta_n$ into itself, one sees from the Frobenius–Perron theory for nonnegative matrices that (17) is solvable. Moreover, using Theorems 3.1 and 3.2, we have

Theorem 4.1. Suppose that $C_{BV}\alpha < 1$ where C_{BV} is the same as in Theorem 3.1. If P has a unique fixed density $f^* \in BV(I^N)$, then

$$\lim_{n\to\infty} \|f_n - f^*\| = 0,$$

$$\|f_n - f^*\|_{BV} = O(\|Q_n f^* - f^*\|_{BV}).$$

Moreover

$$\|f_n - f^*\|_{BV} = o(1), \qquad \text{if} \quad f^* \in W^{1,1}(I^N),$$

$$\|f_n - f^*\|_{BV} = O(h), \qquad \text{if} \quad f^* \in C^2(I^N).$$

Proof. From (11), the definition of the BV-norm, the fact that P and Q_n preserve the L^1-norm of f, and Theorem 3.1, we have

$$\|f_n\|_{BV} = \|Q_n P f_n\|_{BV} \leqslant C_{BV}(\alpha\, \|f_n\|_{BV} + \beta).$$

Hence $\|f_n\|_{BV} \leqslant C_{BV}\beta/(1 - C_{BV}\alpha)$ uniformly, which implies that f_n has a L^1-convergent subsequence which converges to a fixed density of P. Since f^* is the unique fixed density of P, it turns our f_n actually converges to f^* under the L^1-norm. Now from

$$f_n - f^* = Q_n P(f_n - f^*) + Q_n f^* - f^*,$$

we see that

$$\|f_n - f^*\|_{BV} \leqslant \frac{1}{1 - C_{BV}\alpha} (C_{BV}\beta \|f_n - f^*\| + \|Q_n f^* - f^*\|_{BV}).$$

Thus, using a quasi-compactness argument as in ref. 5 and Theorem 3.2, we get the last two conclusions. ▮

4.2. A Numerical Example

We present some numerical experiments with the piecewise linear finite element method for computing fixed densities of a Markov integral operator

$$Pf(x) = \int_0^1 K(x, y)\, f(y)\, dy, \qquad f \in L^1(0, 1), \tag{13}$$

where the stochastic kernel

$$K(x, y) = \frac{ye^{xy}}{e^y - 1}, \tag{14}$$

as compared with Ulam's piecewise constant method. For the simplicity we divide $[0, 1]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$ with length $h = 1/n$. Let f_i be the average value of f over I_i. Then Ulam's scheme is given by

$$Q_h^0 f(x) = \sum_{i=1}^{n} f_i \chi_{I_i}(x), \tag{15}$$

where $\chi_{I_i}(x) = 1$ if $x \in I_i$ and 0 otherwise, and our method uses

$$Q_h^1 f(x) = f_1 e_0^1(x) + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e_i^1(x) + f_n e_n^1(x), \tag{16}$$

where

$$e_i^1(x) = w\left(\frac{x - x_i}{h}\right), \qquad i = 0, 1, \ldots, n$$

with $w(x) = (1 - |x|)\, \chi_{[-1,1]}$.

In the implementation we let $n = 2^r$ with $r = 2, 3, \ldots, L$ for some given L. The integration technique of the trapezoid rule was employed for the evaluation of the matrix representation of $P_h^0 = Q_h^0 P$ and $P_h^1 = Q_h^1 P$ with respect to the density basis $\{\chi_{I_i}/h\}$ and the density basis $\{e_i^1/\|e_i^1\|\}$, respectively. For

Table I. L^1-Norm Errors

number of subinterval	L^1-norm errors	
	piecewise constant method	piecewise linear method
2^2	3.399×10^{-2}	1.564×10^{-2}
2^3	1.702×10^{-2}	4.167×10^{-3}
2^4	8.515×10^{-3}	1.076×10^{-3}
2^5	4.258×10^{-3}	2.733×10^{-4}
2^6	2.129×10^{-3}	6.889×10^{-5}
2^7	1.064×10^{-3}	1.729×10^{-5}
2^8	5.322×10^{-4}	4.333×10^{-6}
2^9	2.611×10^{-4}	1.084×10^{-6}

$j = 0, 1$, because of the integration error, each column of the matrix was normalized so that the resulting matrix $\tilde{P}_h^j$ is a stochastic one. Then the direct iteration was used to find a normalized fixed nonnegative vector v_h^j of $(\tilde{P}_h^j)^T$, starting from the unit positive vector of the same components. The convergence was obtained after a couple of iterations (less than 10 for all dimensions in the computation).

Since the expression of the fixed density f^* of P is unknown, for $j = 0, 1$, we used $\|f_{2h}^j - f_h^j\|$ and $\|f_{2h}^j - f_h^j\|_{BV}$ to estimate the L^1-norm error $\|f^* - f_h^j\|$ and the BV-norm error $\|f^* - f_h^j\|_{BV}$ of f_h^j from the piecewise constant method and the piecewise linear method, respectively.

The computational results from Tables I and II show that for the piecewise linear method, the BV-norm error reduces about the same order as h, which is consistent with our theoretical result. Furthermore, the L^1-norm error reduces at the order of h^2, which can be explained with the fact that

Table II. BV-Norm Errors

number of subinterval	BV-norm errors	
	piecewise constant method	piecewise linear method
2^2	3.059×10^{-1}	1.233×10^{-1}
2^3	2.894×10^{-1}	6.621×10^{-2}
2^4	2.810×10^{-1}	3.432×10^{-2}
2^5	2.768×10^{-1}	1.747×10^{-2}
2^6	2.746×10^{-1}	8.813×10^{-3}
2^7	2.736×10^{-1}	4.426×10^{-3}
2^8	2.730×10^{-1}	2.218×10^{-3}
2^9	2.728×10^{-1}	1.110×10^{-3}

$\|f - Q_h^1 f\| = O(h^2)$. On the other hand, although the piecewise constant method does converge in the L^1-norm, it is not so under the BV-norm since $\bigvee_0^1 (f - Q_h^0 f) \geqslant \bigvee_0^1 f$ in general (see Proposition 3.3 in ref. 5).

ACKNOWLEDGMENTS

The first author was supported in part by a China Bridge fellowship through the Chinese National Science Foundation and the China Bridges International (North America), and the second author was subsidized by the Special Funds for Major State Basic Research Projects and also supported in part by the Chinese National Science Foundation and the Educational Center for Mathematical Research sponsored by the Ministry of Education of China. They also thank the referee for suggestions for a better presentation.

REFERENCES

1. C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, 1993).
2. C. Bose and R. Murray, The exact rate of approximation in Ulam's method, *Disc. Cont. Dyna. Sys.* **7**(1):219–235 (2001).
3. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, 1978).
4. J. Ding and T. Y. Li, Markov finite approximation of Frobenius–Perron operator, *Nonlinear Anal., TMA* **17**(8):759–772 (1991).
5. J. Ding and T. Y. Li, A convergence rate analysis for Markov finite approximations to a class of Frobenius–Perron operators, *Nonlinear Anal., TMA* **31**(5/6):765–777 (1998).
6. J. Ding and A. Zhou, Piecewise linear Markov approximations of Frobenius–Perron operators associated with multi-dimensional transformations, *Nonlinear Anal., TMA* **25**(4):399–408 (1995).
7. S. Foguel, *The Ergodic Theory of Markov Processes* (Van Nostrand Mathematical Studies 21, 1969).
8. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, 1983).
9. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation* (Birkhäuser, 1984).
10. P. Góra and A. Boyarsky, Absolutely continuous invariant measures for piecewise expanding C^2 transformations in R^N, *Israel J. Math.* **67**(3):272–286 (1989).
11. G. Keller, Stochastic stability in some chaotic dynamical systems, *Mon. Math.* **94**:313–353 (1982).
12. A. Lasota and M. Mackey, *Chaos, Fractals, and Noises*, 2nd edn. (Springer-Verlag, 1994).
13. A. Lasota and J. Yorke, On the existence of invariant measures for piecewise monotonic transformations, *Trans. Amer. Math. Soc.* **186**:481–488 (1973).
14. T. Y. Li, Finite approximation for the Frobenius–Perron operator, a solution to Ulam's conjecture, *J. Approx. Theory* **17**:177–186 (1976).
15. R. Murray, Approximation error for invariant density calculations, *Discrete and Cont. Dynam. Sys.* **4**(3):535–557 (1998).
16. M. J. Todd, *The Computation of Fixed points and Applications* (Springer-Verlag, 1976).
17. S. M. Ulam, A Collection of Mathematical Problems, *Interscience Tracts in Pure and Applied Math.* **8** (Interscience, 1960).